Strategically equivalent contests

Dias Da Cruz Steve

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UNIVERSITÉ DU LUXEMBOURG

Faculty of Sciences, Technology and Communication
University of Luxembourg
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Contents

1 Preliminaries

First, just a quick reminder of two important definite integrals. The first one is just an easy computation and for the second one it is possible to get the result from integration by parts.

$$\int_0^\infty \mathrm{e}^{-kx} \;\mathrm{d}x = \frac{1}{k} \qquad \text{if } k > 0. \tag{1}$$

$$\int_0^\infty x\,\mathrm{e}^{-kx} \;\mathrm{d}x = \frac{1}{k^2} \qquad \text{if } k > 0. \tag{2}$$

The density of the uniform distribution is given by

$$f(x) = \frac{1}{b-a}\mathbb{1}_{a \le x \le b} \tag{3}$$

and the density of the exponential distribution is given by

$$f(x) = \lambda\,\mathrm{e}^{-\lambda x}\,\mathbb{1}_{0 \le x} \tag{4}$$

If $f(x)$ is a probability density function of the probability distribution X, then the expectation can be computed by

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x \tag{5}$$

and by the transformation lemma, we can conclude that for a measurable function $g(x)$ of X we get

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,\mathrm{d}x. \tag{6}$$

The bilateral **Laplace transform** (or two-sided Laplace transform) of a function $f(x)$ (or of a random variable X with density function f) is given by

$$F(r) = \int_{-\infty}^{\infty} \mathrm{e}^{-rx}\, f(x)\,\mathrm{d}x = \mathrm{E}[\mathrm{e}^{-rX}] = \mathcal{L}\{f(x)\}. \tag{7}$$

Further, we know the following property of the Laplace transform

$$-F'(r) = \mathcal{L}\{x f(x)\}. \tag{8}$$

2 Game theory and strategically equivalent contests

This section will be a short introduction to game theory and discuss, why we are interested in a very specific differential equation, which will be analysed and solved in the continuation of this paper.

2.1 Introduction

Assume that we have two players who want to compete for the same price. For the sake of simplicity, let's assume that we have two politicians **A** and **B** , who want to win the election. Hence, their price is the votes of the population. Denote by **V** the total value of the price, in our case, the total amount of votes they can get. Each politician can invest a certain amount of effort into his campaign (money he wants to invest into advertising, time he wants to spend on events, effort he wants to spend on lobbing, etc.). Denote by x_A the amount of effort of politician **A** and by x_B the amount of effort of politician **B**, which is put into their campaign respectively. Hence, the total effort put into winning the election is $x_A + x_B$. In an ideal world, this would mean that politician A would get a proportion $\frac{x_A}{x_A+x_B}$ of the total votes, whereas politician B would get a proportion $\frac{x_B}{x_A+x_B}$ of the total votes. Depending on the strategy each politician is using, one could define the costs per unit (c_a and c_b) of the invested effort for each politician. Consequently, by taken into account the actual costs of the invested effort, the win, profit or payoff for each politician would then be given by

$$\begin{aligned}\pi_A &= V\frac{x_A}{x_A + x_B} - c_a x_A \\ \pi_B &= V\frac{x_B}{x_A + x_B} - c_b x_B.\end{aligned}$$

Since each politician only gets a proportion of the prize, we call this situation a **proportional contest**.

For the next situation, we have player **A** and player **B** competing for the same lottery prize **V**. Either way, the prize will be paid to one of the two players (it is not possible, that nobody will win the prize). Denote by x_A the amount of effort of player **A** and by x_B the amount of effort of player **B**, which is put into winning the prize V. In this case, player A has a probability $\frac{x_A}{x_A+x_B}$ of winning the prize, whereas player B has a probability $1 - \frac{x_A}{x_A+x_B} = \frac{x_B}{x_A+x_B}$

of winning the same prize. Hence, the payoff is given by

$$\pi_A = (V - x_A)\frac{x_A}{x_A + x_B}$$
$$\pi_B = (V - x_B)\frac{x_B}{x_A + x_B}.$$

It is important to notice that one of the players will win nothing and hence will only have a loss. Hence, player A loses with a probability of $1 - \frac{x_A}{x_A+x_B} = \frac{x_B}{x_A+x_B}$ an amount of $-x_A$ and player B loses with a probability of $1 - \frac{x_B}{x_A+x_B} = \frac{x_A}{x_A+x_B}$ an amount of $-x_B$, because the prize would be $V = 0$. This stands in contrast with the previous situation, where each player gets a proportion of the prize. Since in this situation only on player can win, we call it a **lottery contest**.

In both situations x_i is the amount of effort put into winning the same risky prize. However, both players do not choose the amount of effort independently of the other player. Hence, if I am player A and I want to compete against player B, I need to choose my amount of effort by taking into account how much effort I expect player B to invest into winning the same prize as me. Further, it is important to notice that I need to choose my amount of effort with regard to the costs, the total amount I can win and the total loss I can suffer.

A player is said to be **risk-averse**, if he would rather take a payment (certain win), which is less than the possible win of a risky prize V instead of gambling for that risky prize. For instance, if the risky prize is 500 dollars and the certain payment is 250 dollars, the risk-averse player would take the certain win an not gamble for the uncertain prize. A player is said to be **risk-neutral** if he is indifferent between choosing the certain payment or going for the uncertain prize.

If x represents the amount of the risky prize which a player can win, then the player has a utility function $u(x)$, which describes the resulting utility with respect to the possible prize. In utility theory, time is not taken into account for the utility function.

In the absence of risk aversion and risky rent or if players are risk-neutral, then the lottery contest and the proportional contest are strategically equivalent. Intuitively, one might expect that a risk-averse player under risky rents should invest more effort into a proportional contest than into a lottery contest. The omnipresent differential equation of this paper is the solution to the question in which cases both strategies are equivalent for risk-averse players under a risky rent.

An informal definition of the Nash equilibrium (named after John Forbes Nash) is the following: If each player knows the strategy of all other players

and if each player can not improve his result by changing his strategy (by knowing the strategy of each other player), then the strategies of all players together are in a Nash equilibrium.

In the following, these situations are theorized and generalized by the following framework.

2.2 Framework

Two risk-averse players compete for a risky prize V. Each player i chooses an effort level x_i given the expected effort level of his rival x_j. The vector of efforts (x_i, x_j) determines which player will receive which prize. We use the following functional for $p_i(x_i, x_j)$ ("Contest success function")

$$p_i = \frac{x_i}{x_i + x_j}.$$

p_i is either interpreted as a share of the prize (Proportional contest) or as a win probability (Lottery contest). For both cases, p_i equals player i's effort relative to the total effort. Further, players are expected utility maximizer. This means that each player tries to win the maximal possible amount with respect to the costs and the expected effort put into account by the rival. In the following, $u(x)$ represents the utility function. For the **lottery contest**, we can win with probability p_i an amount of $V - x_i$ and lose with probability $1 - p_i$ an amount of $-x_i$. The payoff is given by

$$\pi_i = (V - x_i, -x_i; p_i, 1 - p_i)$$

and the expected utility is

$$\begin{aligned} \mathrm{E}[u(\pi_i(x))] &= p_i u(V - x_i) + (1 - p_i) u(-x_i) \\ &= \frac{x_i}{x_i + x_j} u(V - x_i) + \frac{x_j}{x_i + x_j} u(-x_i), \end{aligned}$$

whereas for the **proportional contest** you win a share of the Price. The payoff is given by

$$\pi_i = (p_i V - x_i; 1),$$

and the expected utility is

$$\mathrm{E}[u(\pi_i(x))] = 1 * u(p_i V - x_i) = u\left(\frac{x_i}{x_i + x_j} V - x_i\right).$$

By definition, risk-averse players have constant absolute risk-averse (CARA) preferences, which is represented by the following negative exponential utility function:

$$u(\pi_i) = -\,\mathrm{e}^{-r\pi_i}.$$

where $r > 0$ represents the **coefficient of absolute risk aversion**, which is defined as $(r = -\frac{u''}{u'})$, where $u(x)$ is the utility function. If players are risk-neutral, the expected utility maximizer is given by $u(x) = x$. In this case, one can easily prove that both contest are equivalent, since

$$\begin{aligned}
\mathrm{E}_L[u(\pi_i(x))] &= \frac{x_i}{x_i + x_j}(V - x_i) + \frac{x_j}{x_i + x_j}(-x_i) \\
&= \frac{x_i}{x_i + x_j}V - x_i\Big(\frac{x_i}{x_i + x_j} + \frac{x_j}{x_i + x_j}\Big) \\
&= \frac{x_i}{x_i + x_j}V - x_i \\
&= \mathrm{E}_P[u(\pi_i(x))].
\end{aligned}$$

In the following, F represents the Laplace transformation of f.

2.3 P contest

For player i, the expected utility with uncertain payoff $\pi_i = p_i(V + \epsilon) - x_i$ is equal to:

$$\begin{aligned}
\mathrm{E}[u(\pi_i)] &= \mathrm{E}[-\,\mathrm{e}^{-r\pi_i}] \\
&= \int_{-\infty}^{\infty} (-\,\mathrm{e}^{-r(p_i\epsilon + p_iV - x_i)}) f(\epsilon)\,\mathrm{d}\epsilon \\
&= -\,\mathrm{e}^{-r(p_iV - x_i)} \int_{-\infty}^{\infty} (-\,\mathrm{e}^{-rp_i\epsilon}) f(\epsilon)\,\mathrm{d}\epsilon \\
&= -\,\mathrm{e}^{rx_i - \frac{rx_i}{x_i + x_j}V}\, F\left(\frac{-rx_i}{x_i + x_j}\right).
\end{aligned}$$

The first-order condition for a symmetrical Nash equilibrium is

$$\frac{\partial}{\partial u_i}\Big(\mathrm{E}[u(\pi_i)]\Big) = 0,$$

Computing the partial derivative yields

$$\begin{aligned}
\frac{\partial}{\partial x_i}\Big(\mathrm{E}[u(\pi_i)]\Big) &= \frac{\partial}{\partial x_i}\left(-\,\mathrm{e}^{rx_i - \frac{rx_i}{x_i + x_j}V}\, F\Big(\frac{-rx_i}{x_i + x_j}\Big)\right) \\
&= -\left(-\frac{rV}{x_i + x_j} + \frac{rx_iV}{(x_i + x_j)^2} + r\right) \mathrm{e}^{rx_i - \frac{rx_iV}{x_i + x_j}}\, F\Big(-\frac{rx_i}{x_i + x_j}\Big) \\
&\quad - \left(\frac{rx_i}{(x_i + x_j)^2} - \frac{r}{x_i + x_j}\right) \mathrm{e}^{rx_i - \frac{rx_iV}{x_i + x_j}}\, F'\Big(-\frac{rx_i}{x_i + x_j}\Big),
\end{aligned}$$

which then for the symmetrical equilibrium case $x_i = x \; \forall i$ (everyone chooses the same effort level) yields

$$\begin{aligned}
\frac{\partial}{\partial x_i}\Big(\mathrm{E}[u(\pi_i)]\Big) &= -\left(-\frac{rV}{x+x} + \frac{rxV}{(x+x)^2} + r\right) \mathrm{e}^{rx - \frac{rxV}{x+x}} F\Big(-\frac{rx}{x+x}\Big) \\
&\quad - \left(\frac{rx}{(x+x)^2} - \frac{r}{x+x}\right) \mathrm{e}^{rx - \frac{rxV}{x+x}} F'\Big(-\frac{rx}{x+x}\Big) \\
&= -\left(-\frac{rV}{2x} + \frac{rV}{4x} + r\right) \mathrm{e}^{rx - \frac{rV}{2}} F\Big(-\frac{r}{2}\Big) - \Big(\frac{r}{4x} - \frac{r}{2x}\Big) \mathrm{e}^{rx - \frac{rV}{2}} F'\Big(-\frac{r}{2}\Big) \\
&= -r\left(-\frac{V}{4x} + 1\right) \mathrm{e}^{rx - \frac{rV}{2}} F\Big(-\frac{r}{2}\Big) + \frac{r}{4x} \mathrm{e}^{rx - \frac{rV}{2}} F'\Big(-\frac{r}{2}\Big) \\
&= 0.
\end{aligned}$$

Hence we get the condition

$$\begin{aligned}
&-(4x - V)F(-\frac{r}{2}) + F'(-\frac{r}{2}) = 0 \\
\iff\; &(4x - V)F(-\frac{r}{2}) = F'(-\frac{r}{2}) \\
\iff\; &4x = \frac{F'(-\frac{r}{2})}{F(-\frac{r}{2})} + V
\end{aligned}$$

which leads to a candidate expression for the equilibrium effort

$$X_S = \frac{1}{4}\left(\frac{F'(-\frac{r}{2})}{F(-\frac{r}{2})} + V\right).$$

2.4 Lottery contest

For player i, the expected utility with uncertain payoff $\pi_i = (V + \epsilon) - x_i$ is equal to:

$$\begin{aligned}
\mathrm{E}[u(\pi_i)] &= \frac{x_i}{x_i + x_j}\left(\int_{-\infty}^{\infty} (-\,\mathrm{e}^{-r(V+\epsilon-x_i)}) f(\epsilon)\,\mathrm{d}u\right) + \frac{x_j}{x_i + x_j}\left(\int_{-\infty}^{\infty} (-\,\mathrm{e}^{rx_i}) f(\epsilon)\,\mathrm{d}\epsilon\right) \\
&= -\frac{x_i\,\mathrm{e}^{rx_i - rV}}{x_i + x_j}\left(\int_{-\infty}^{\infty} \mathrm{e}^{-r\epsilon} f(\epsilon)\,\mathrm{d}\epsilon\right) + \frac{x_j}{x_i + x_j}(-\,\mathrm{e}^{rx_i})\left(\int_{-\infty}^{\infty} f(\epsilon)\,\mathrm{d}\epsilon\right) \\
&= -\frac{x_i\,\mathrm{e}^{rx_i - rV}}{x_i + x_j} F(-r) - \frac{x_j\,\mathrm{e}^{rx_i}}{x_i + x_j},
\end{aligned}$$

which then implies

$$\begin{aligned}
\frac{\partial}{\partial x_i}\Big(\mathrm{E}[u(\pi_i)]\Big) &= \frac{\partial}{\partial x_i}\left(-\frac{x_i\,\mathrm{e}^{rx_i-rV}}{x_i+x_j}F(-r)-\frac{x_j\,\mathrm{e}^{rx_i}}{x_i+x_j}\right)\\
&= -\frac{rx_i\,\mathrm{e}^{rx_i-rV}}{x_i+x_j}F(-r)+\frac{x_i\,\mathrm{e}^{rx_i-rV}}{(x_i+x_j)^2}F(-r)-\frac{\mathrm{e}^{rx_i-rV}}{x_i+x_j}F(-r)\\
&\qquad -\frac{rx_j\,\mathrm{e}^{rx_i}}{x_i+x_j}+\frac{x_j\,\mathrm{e}^{rx_i}}{(x_i+x_j)^2}.
\end{aligned}$$

For the symmetrical equilibrium case $x_i = x\ \forall i$ and the symmetrical Nash equilibrium we get

$$\begin{aligned}
\frac{\partial}{\partial x_i}\Big(\mathrm{E}[u(\pi_i)]\Big) &= -\frac{rx\,\mathrm{e}^{rx-rV}}{x+x}F(-r)+\frac{x\,\mathrm{e}^{rx-rV}}{(x+x)^2}F(-r)-\frac{\mathrm{e}^{rx-rV}}{x+x}F(-r)-\frac{rx\,\mathrm{e}^{rx}}{x+x}+\frac{x\,\mathrm{e}^{rx}}{(x+x)^2}\\
&= -\frac{r\,\mathrm{e}^{rx-rV}}{2}F(-r)+\frac{\mathrm{e}^{rx-rV}}{4x}F(-r)-\frac{\mathrm{e}^{rx-rV}}{2x}F(-r)-\frac{r\,\mathrm{e}^{rx}}{2}+\frac{\mathrm{e}^{rx}}{4x}\\
&= -\frac{1}{2}r\,\mathrm{e}^{rx}\,\mathrm{e}^{-rV}\,F(-r)-\frac{1}{4x}\,\mathrm{e}^{rx}\,\mathrm{e}^{-rV}\,F(-r)-\frac{1}{2}r\,\mathrm{e}^{rx}+\frac{1}{4x}\,\mathrm{e}^{rx}\\
&= 0.
\end{aligned}$$

After simplification we get

$$-\mathrm{e}^{-rV}\,F(-r)-2xr\,\mathrm{e}^{-rV}\,F(-r)-2xr+1=0,$$

which can be transformed into

$$X_L=\frac{1-\mathrm{e}^{-rV}\,F(-r)}{2r(1+\mathrm{e}^{-rV}\,F(-r))}.$$

2.5 Conclusion

By [2] we can finally conclude by applying the following definition:

Definition 1. Contests are effort equivalent if they result in the same equilibrium efforts.

Thus, we get the equality

$$\frac{1}{4}\left(\frac{F'(-\frac{r}{2})}{F(-\frac{r}{2})}+V\right)=\frac{1-\mathrm{e}^{-rV}\,F(-r)}{2r(1+\mathrm{e}^{-rV}\,F(-r))},$$

and we can conclude by setting $V=0$ (the entire benefit is put to a risk) that the equation is

$$\frac{1}{4}\left(\frac{F'(-\frac{r}{2})}{F(-\frac{r}{2})}\right)=\frac{1-F'(-r)}{2r(1+F'(-r))},$$

which is equivalent to (9). This means, that the solution to this differential equation gives us the cases where the lottery contest and the proportional contest are equivalent for risk-averse players under risky rent (since we put the entire benefit to a risk).

We can also give an interpretation for examples (3.2) and (3.3). For the normal distribution and for the uniform distribution, we see that a risk-averse player under risky rent should go for the proportional contest and not for the lottery contest.

3 Integral equation

This paper will focus on the following equation for which we want to find explicit solutions as well as numerical approximations of solutions.

$$\frac{1}{4}\frac{\mathrm{E}[X\exp(-\frac{r}{2}X)]}{\mathrm{E}[\exp(-\frac{r}{2}X)]} = \frac{1-\mathrm{E}[X\exp(-rX)]}{2r(1+\mathrm{E}[\exp(-rX)])}, \tag{9}$$

whereas we will try to find solutions for the equivalent equation:

$$\begin{aligned} &\frac{1}{4}\frac{\mathrm{E}[X\exp(-\frac{r}{2}X)]}{\mathrm{E}[\exp(-\frac{r}{2}X)]} = \frac{1-\mathrm{E}[X\exp(-rX)]}{2r(1+\mathrm{E}[\exp(-rX)])}, \\ \iff\ &\frac{1}{4}\frac{\int_{\mathbb{R}} x\,\mathrm{e}^{-\frac{r}{2}x}\,f(x)\,\mathrm{d}x}{\int_{\mathbb{R}}\mathrm{e}^{-\frac{r}{2}x}\,f(x)\,\mathrm{d}x} = \frac{1-\int_{\mathbb{R}}\mathrm{e}^{-rx}\,f(x)\,\mathrm{d}x}{2r(1+\int_{\mathbb{R}}\mathrm{e}^{-rx}\,f(x)\,\mathrm{d}x)} \\ &\iff X_S = X_L \end{aligned} \tag{10}$$

By multiplying both sides of (10) by their denominators, we can easily transform the equality into an integral equation

$$2r\int_{\mathbb{R}} x\,\mathrm{e}^{-\frac{r}{2}x}\,f(x)\,\mathrm{d}x\left(1+\int_{\mathbb{R}}\mathrm{e}^{-rx}\,f(x)\,\mathrm{d}x\right) = 4\int_{\mathbb{R}}\mathrm{e}^{-\frac{r}{2}x}\,f(x)\,\mathrm{d}x\left(1-\int_{\mathbb{R}}\mathrm{e}^{-rx}\,f(x)\,\mathrm{d}x\right)$$

I will present a possible solution for the presented equation as well as two examples, which do not work.

Example 3.1. *For the first example, let us take the density of the exponential distribution and let us show that it is actually a solution for the equation. By (1), we immediately conclude that the factors of the left hand-side of (10) are equal to*

$$\int_{\mathbb{R}}\mathrm{e}^{-\frac{r}{2}x}\,f(x)\,\mathrm{d}x = \int_0^\infty \mathrm{e}^{-\frac{r}{2}x}\,\lambda\,\mathrm{e}^{-\lambda x}\,\mathrm{d}x = \lambda\int_0^\infty \mathrm{e}^{-(\frac{r}{2}+\lambda)x}\,\mathrm{d}x = \frac{\lambda}{\frac{r}{2}+\lambda},$$

$$\int_{\mathbb{R}} x\,\mathrm{e}^{-\frac{r}{2}x}\, f(x)\,\mathrm{d}x = \int_0^\infty x\,\mathrm{e}^{-\frac{r}{2}x}\,\lambda\,\mathrm{e}^{-\lambda x}\ \mathrm{d}x = \lambda \int_0^\infty x\,\mathrm{e}^{-(\frac{r}{2}+\lambda)x}\ \mathrm{d}x = \frac{\lambda}{(\frac{r}{2}+\lambda)^2},$$

which then implies that

$$X_S = \frac{1}{4}\frac{\frac{\lambda}{(\frac{r}{2}+\lambda)^2}}{\frac{\lambda}{\frac{r}{2}+\lambda}} = \frac{1}{2r+4\lambda}$$

By the same procedure, we get that the factors of the right hand-side of (10) are equal to

$$\int_{\mathbb{R}} \mathrm{e}^{-rx}\, f(x)\,\mathrm{d}x = \int_0^\infty \mathrm{e}^{-rx}\,\lambda\,\mathrm{e}^{-\lambda x}\ \mathrm{d}x = \lambda \int_0^\infty \mathrm{e}^{-(r+\lambda)x}\ \mathrm{d}x = \frac{\lambda}{r+\lambda},$$

which then implies that

$$X_L = \frac{1-\frac{\lambda}{r+\lambda}}{2r(1+\frac{\lambda}{r+\lambda})} = \frac{\frac{r}{r+\lambda}}{2r\frac{r+2\lambda}{r+\lambda}} = \frac{1}{2r+4\lambda}$$

So finally we get that $X_S = X_L$.

Example 3.2. *For the second example we will take a density, which does not satisfy the equation. For simplicity we will take the standard normal distribution. We immediately conclude that the factors of the left hand-side of (10) are equal to*

$$\int_{\mathbb{R}} \mathrm{e}^{-\frac{r}{2}x}\, f(x)\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{e}^{-\frac{r}{2}x}\,\mathrm{e}^{-\frac{x^2}{2}}\ \mathrm{d}x = \mathrm{e}^{\frac{r^2}{8}},$$

and

$$\int_{\mathbb{R}} x\,\mathrm{e}^{-\frac{r}{2}x}\, f(x)\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} x\,\mathrm{e}^{-\frac{r}{2}x}\,\mathrm{e}^{-\frac{x^2}{2}}\ \mathrm{d}x = -\frac{r}{2}\,\mathrm{e}^{\frac{r^2}{8}}\ .$$

which then implies that

$$X_S = \frac{1}{4}\frac{-\frac{r}{2}\,\mathrm{e}^{\frac{r^2}{8}}}{\mathrm{e}^{\frac{r^2}{8}}} = -\frac{r}{8}.$$

By the same procedure, we get that the right side of (10) is equal to

$$\int_{\mathbb{R}} \mathrm{e}^{-rx}\, f(x)\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{e}^{-rx}\,\mathrm{e}^{-\frac{x^2}{2}}\ \mathrm{d}x = \mathrm{e}^{\frac{r^2}{2}},$$

which then implies that

$$X_L = \frac{1-\mathrm{e}^{\frac{r^2}{2}}}{2r(1+\mathrm{e}^{\frac{r^2}{2}})} = -\frac{\tanh\frac{r^2}{4}}{2r}.$$

Clearly, we have that

$$X_S \neq X_L \qquad \textit{for all } r > 0. \tag{11}$$

However, we can even find out more. Since $r \geq 0$ *we get that*

$$X_S \geq X_L \qquad \textit{for all } r \geq 0.$$

To prove this, we are going to use the well known inequality

$$\mathrm{e}^x \geq 1 + x$$

Starting with the inequality, we will prove that it is always satisfied by minoring it:

$$\begin{aligned}
&X_S - X_L \geq 0 \\
&\iff \frac{1-\mathrm{e}^{\frac{r^2}{2}}}{2r(1+\mathrm{e}^{\frac{r^2}{2}})} + \frac{r}{8} \geq 0 \\
&\iff 4(1-\mathrm{e}^{\frac{r^2}{2}})r(r(1+\mathrm{e}^{\frac{r^2}{2}})) \geq 0 \\
&\iff 4 - 4\,\mathrm{e}^{\frac{r^2}{2}} + r^2 + r^2\,\mathrm{e}^{\frac{r^2}{2}} \geq 0 \\
&\iff 4 - 4(1+\frac{r^2}{2}) + r^2 + r^2(1+\frac{r^2}{2}) \geq 0 \\
&\iff 4 - 4 - 2r^2 + r^2 + r^2 + \frac{r^4}{2} \geq 0 \\
&\iff \frac{r^4}{2} \geq 0,
\end{aligned}$$

which is true for all $r \geq 0$ *and equality is only obtained for* $r = 0$.

Example 3.3. *A second counter-example, let us take the density of the uniform distribution. We immediately conclude that the left side of (10) is equal to*

$$\int_{\mathbb{R}} \mathrm{e}^{-\frac{r}{2}x} f(x)\,\mathrm{d}x = \frac{1}{b-a}\int_a^b \mathrm{e}^{-\frac{r}{2}x}\,\mathrm{d}x = \frac{2}{r(b-a)}(\mathrm{e}^{-\frac{ar}{2}} - \mathrm{e}^{-\frac{br}{2}}),$$

$$\int_{\mathbb{R}} x\,\mathrm{e}^{-\frac{r}{2}x} f(x)\,\mathrm{d}x = \frac{1}{b-a}\int_a^b x\,\mathrm{e}^{-\frac{r}{2}x}\,\mathrm{d}x = \frac{2}{r^2(b-a)}(\mathrm{e}^{-\frac{ar}{2}}(ar+2) - \mathrm{e}^{-\frac{br}{2}}(br+2)),$$

which then implies that

$$X_S = \frac{1}{4}\frac{\frac{2}{r^2(b-a)}(\mathrm{e}^{-\frac{ar}{2}}(ar+2) - \mathrm{e}^{-\frac{br}{2}}(br+2))}{\frac{2}{r(b-a)}(\mathrm{e}^{-\frac{ar}{2}} - \mathrm{e}^{-\frac{br}{2}})} = \frac{1}{4r}\frac{(\mathrm{e}^{-\frac{ar}{2}}(ar+2) - \mathrm{e}^{-\frac{br}{2}}(br+2))}{(\mathrm{e}^{-\frac{ar}{2}} - \mathrm{e}^{-\frac{br}{2}})}$$

By the same procedure, we get that the right side of (10) is equal to

$$\int_{\mathbb{R}} \mathrm{e}^{-rx} f(x)\,\mathrm{d}x = \frac{1}{b-a}\int_a^b \mathrm{e}^{-rx}\,\mathrm{d}x = \frac{1}{b-a}\frac{\mathrm{e}^{-ar}-\mathrm{e}^{-br}}{r},$$

which then implies that

$$X_L = \frac{1-\frac{1}{b-a}\frac{\mathrm{e}^{-ar}-\mathrm{e}^{-br}}{r}}{2r(1+\frac{1}{b-a}\frac{\mathrm{e}^{-ar}-\mathrm{e}^{-br}}{r})} = \frac{r(a-b)\,\mathrm{e}^{r(a+b)}-\mathrm{e}^{ar}+\mathrm{e}^{br}}{2r(r(a-b)\,\mathrm{e}^{r(a+b)}+\mathrm{e}^{ar}-\mathrm{e}^{br})}.$$

For simplicity, let us take $a=0$ and $b=1$. This yields

$$X_S = \frac{2-\mathrm{e}^{-\frac{r}{2}}(r+2)}{4r(1-\mathrm{e}^{-\frac{r}{2}})}$$

$$X_L = \frac{-r\,\mathrm{e}^r-1+\mathrm{e}^r}{2r(-r\,\mathrm{e}^r+1-\mathrm{e}^r)} = \frac{\mathrm{e}^r(1-r)-1}{2r(1-\mathrm{e}^r(1+r))}$$

Using the fact that

$$1-\mathrm{e}^{-\frac{r}{2}} > 0 \qquad \text{for all } r>0$$

$$1-\mathrm{e}^r(1+r) < 0 \qquad \text{for all } r>0,$$

we are going to prove that $X_S > X_L$ for all $r>0$ by the following:

$$\begin{aligned}
&X_S - X_L \geq 0\\
\iff\ &\frac{2-\mathrm{e}^{-\frac{r}{2}}(r+2)}{4r(1-\mathrm{e}^{-\frac{r}{2}})} - \frac{\mathrm{e}^r(1-r)-1}{2r(1-\mathrm{e}^r(1+r))} \geq 0\\
\iff\ &2\,\mathrm{e}^{\frac{r}{2}}\,r^3 - 2\,\mathrm{e}^{-\frac{r}{2}}\,r^2 + 2\,\mathrm{e}^{\frac{r}{2}}\,r^2 - 8\,\mathrm{e}^{-\frac{r}{2}}\,r + 8\,\mathrm{e}^{\frac{r}{2}}\,r - 8\,\mathrm{e}^r\,r + 8r \leq 0\\
\iff\ &2\,\mathrm{e}^r\,r^3 - 2r^2 + 2\,\mathrm{e}^r\,r^2 - 8r + 8\,\mathrm{e}^r\,r - 8\,\mathrm{e}^{\frac{3r}{2}}\,r + 8r\,\mathrm{e}^{\frac{r}{2}} \leq 0\\
\iff\ &4\,\mathrm{e}^{\frac{r}{2}} - r - 4\,\mathrm{e}^{\frac{3r}{2}} + 4\,\mathrm{e}^r + r^2\,\mathrm{e}^r + r\,\mathrm{e}^r - 4 \leq 0,
\end{aligned}$$

which is true for all $r>0$ by computation in (5.2).

4 Differential equation

By applying the bilateral Laplace transform to the factors of our differential equation, we get

$$\int_{\mathbb{R}} \mathrm{e}^{-rx} f(x)\,\mathrm{d}x = F(r)$$

$$\int_{\mathbb{R}} \mathrm{e}^{-\frac{r}{2}x} f(x)\,\mathrm{d}x = F\Big(\frac{r}{2}\Big)$$

$$\int_{\mathbb{R}} x\,\mathrm{e}^{-\frac{r}{2}x} f(x)\,\mathrm{d}x = -F'\Big(\frac{r}{2}\Big).$$

Using the bilateral Laplace transform,the last property of the preliminaries and the just expressed transformations, we can express the integral equation as a differential equation.

$$-2rF'\left(\frac{r}{2}\right)(1+F(r)) = 4F\left(\frac{r}{2}\right)(1-F(r)). \tag{12}$$

$$-\frac{1}{4}\frac{F'(\frac{r}{2})}{F(\frac{r}{2})} = \frac{1-F(r)}{2r(1+F(r))}. \tag{13}$$

We can transform this last equation and get

$$\begin{aligned}
& -2rF'\left(\frac{r}{2}\right)(1+F(r)) = 4F\left(\frac{r}{2}\right)(1-F(r)) \\
\iff & -2r\frac{F'(\frac{r}{2})}{F(\frac{r}{2})} = 4\frac{1-F(r)}{1+F(r)} \\
\iff & -\frac{1}{4}\frac{F'(\frac{r}{2})}{F(\frac{r}{2})} = \frac{1-F(r)}{2r(1+F(r))} \\
\implies & X_S = -\frac{1}{4}\frac{F'(\frac{r}{2})}{F(\frac{r}{2})} \\
\implies & X_L = \frac{1-F(r)}{2r(1+F(r))}
\end{aligned}$$

Further, we can transform the equation differently and apply separation of variables, which finally yields

$$\begin{aligned}
& -2rF'\left(\frac{r}{2}\right)(1+F(r)) = 4F\left(\frac{r}{2}\right)(1-F(r)) \\
\iff & \frac{F'(\frac{r}{2})}{F(\frac{r}{2})} = -\frac{2}{r}\frac{1-F(r)}{1+F(r)} \\
\iff & \ln F\left(\frac{x}{2}\right) = -\int_0^x \frac{2}{r}\frac{1-F(r)}{1+F(r)}\,\mathrm{d}r
\end{aligned}$$

Example 4.1. *A simple example to illustrate how one could proceed in finding a solution by starting from a differential equation. Suppose that we want to find a solution under the form*

$$F(r) = a + br + cr^2,$$

for constants $a, b, c \in \mathbb{R}$. First, let us compute the inverse Laplace transform of $F(r)$ and verify the density condition.

$$\mathcal{L}^{-1}\{F(r)\}(x) = \mathcal{L}^{-1}\{a + br + cr^2\}(x) = a\delta(x) + b\delta'(x) + c\delta''(x) = f(x),$$

where $\delta(x)$ is the Dirac delta function. By definition, the Dirac delta function satisfies

$$\delta(x) = \begin{cases} +\infty, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1.$$

In order for being a density, we need to have

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1.$$

Hence,

$$\begin{aligned}
\int_{-\infty}^{\infty} a\delta(x) + b\delta'(x) + c\delta''(x)\,\mathrm{d}x &= a\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x + b\int_{-\infty}^{\infty} \delta'(x)\,\mathrm{d}x + c\int_{-\infty}^{\infty} \delta''(x)\,\mathrm{d}x \\
&= a * 1 + b[\delta(x)]_{-\infty}^{\infty} + c[\delta'(x)]_{-\infty}^{\infty} \\
&= a + b(0-0) + c(0-0) \\
&= a \\
&= 1,
\end{aligned}$$

which then implies, that we only get a condition for our constant a and that the density condition will not restrict the constants b and c. Hence, our solution needs to be under the form of

$$F(r) = 1 + br + cr^2.$$

Now we can check when $F(r)$ is a solution for our differential equation (it was easier to first check the density condition, since then, our differential equation will be easier to solve). After computation and simplification (see computation in Matlab in example (5.1), we get the condition

$$-cr^3(b + cr) = 0.$$

Thus, we need to have $c = 0$ and $b \in \mathbb{R}$ which finally concludes that the solution has to be

$$F(r) = 1 + br$$

and hence

$$\mathcal{L}^{-1}\{F(r)\}(x) = f(x) = \delta(x) + b\delta'(x).$$

5 Numerical approach

5.1 Introduction

In this section we will try to find numerical solutions for our equation (9). Since it is easier to approximate a solution for a differential equation, we are going to solve the equation

$$-2rF'\left(\frac{r}{2}\right)(1+F(r)) = 4F\left(\frac{r}{2}\right)(1-F(r)) \tag{14}$$

using MATLAB. Since we are solving a differential equation, we will need to have an initial condition. For this, let's check what happends at $F(0)$. Remember that we have that $F(r) = \int_{-\infty}^{\infty} \mathrm{e}^{-rx} f(x)\,\mathrm{d}x$ and hence we get

$$\begin{aligned} F(0) &= \int_{-\infty}^{\infty} \mathrm{e}^{-0x} f(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x \\ &= 1, \end{aligned} \tag{15}$$

which needs to be equal to 1, since we are looking for densities. To begin with, we want to try out the same example as we did in example (3.1).

```
% define the variables as positive parameters
syms x a positive
% define the function with respect to the variable x
g = @(x) a*exp(-a*x)
% compute the laplace transform of g(x)
f = laplace(g,x,r)
```

which then gives us as a result $f(r) = \frac{a}{a+r}$. We are now at a situation, which is equivalent to assume, that the computation for a solution of (14) has provided us the function $f(r)$. So it remains to check, if $f(r)$ is a density (remember that $r > 0$). So we need to compute

```
% compute the inverse laplace of f(r)
h = ilaplace(f, r, x)
% calculate the integral from 0 to infinity (since r>0)
k = int(h, 0, Inf)
```

which tells us, that the integral is always 1. Hence, the result is correct. In a more compact way, one can compute

```
% compute the example in one single line and
% store the different conditions in different variables
[solx,sola, params, conds]
= solve((int(ilaplace(f, r, x), 0, Inf))==1,[x,a],'ReturnConditions', true)
```

which gives the solutions

```
% solution for x
solx = y
% solution for a
sola = u
% the different parameters
params = [ u, y]
% the condition on the parameters
conds = u in Dom::Interval(0, Inf) & y in Dom::Interval(0, Inf)
```

which is exactly the condition that $x > 0$ and $a > 0$. Hence, the result is $a\,\mathrm{e}^{-ax}$ with the just mentioned contitions. Since we always need to check, if the solutions, which we get from (14), are also a density, we now have a possibility to check this by the previous computation (since it provides us with additional constrains on the parameters). Now, to solve a differential equation in MATLAB, one could proceed as follows (simplified example of our problem)

```
% define r as a positive variable
syms r positive
% symbolic representation of F
syms F(r)
% solve the differential equation
dsolve(-2*r*diff(F,r)*(1+F(r))-4*F(r)*(1-F(r))==0, F(0)==1)
```

where the second condition is the initial condition, which was explained ealier. This procedure then would provide the following answers

$$
\begin{array}{l}
1 \\
\dfrac{r^2\,\mathrm{e}^{C_2}}{2} + \dfrac{r\,\mathrm{e}^{\frac{C_2}{2}}\,(\mathrm{e}^{C_2}\,r^2+4)^{\frac{1}{2}}}{2} + 1 \\
\dfrac{r^2\,\mathrm{e}^{C_2}}{2} - \dfrac{r\,\mathrm{e}^{\frac{C_2}{2}}\,(\mathrm{e}^{C_2}\,r^2+4)^{\frac{1}{2}}}{2} + 1
\end{array}
$$

As a short remark, one needs to pay attention to the fact that we evaluate in our differential equation the factor $F'(\frac{x}{2})$ and not $(F(\frac{x}{2}))'$. Hence, we need to compute in Matlab

```
% correct differentiation and evaluation
```

```
subs(diff(f(x)),x,x/2)
% WRONG
diff(f(x/2),x)
```

Example 5.1. *As an example, we can try to solve the equation in example (4.1).*

```
% define variables
syms a b c x
% define function
f = @(x) 1+b*x+c*x^2
% the condition for our differential equation after simplification
simplify(-2*x*subs(diff(f(x)),x,x/2)*(1+f(x))-4*f(x/2)*(1-f(x)))
% or to get the conditions immidiately
simplify(-2*x*subs(diff(f(x)),x,x/2)*(1+f(x))-4*f(x/2)*(1-f(x))==0)
```

This provides the equation

$$-cx^3(b+cx)=0$$

and the conditions

$$b+cx=0|c=0|x=0$$

Example 5.2. *To solve the inequality in example (3.3) we can do the following.*

```
% define variable
syms r positive
% solve numerically the equation
vpasolve(4*exp(r/2) - r - 4*exp((3*r)/2)
           + 4*exp(r) + r^2*exp(r) + r*exp(r) - 4 == 0)
% substitute and calculate r=1 in expression
subs(4*exp(r/2) - r - 4*exp((3*r)/2)
           + 4*exp(r) + r^2*exp(r) + r*exp(r) - 4,r,1)
% get the numerical value of the previous statement
eval(ans)
```

The only numerical solution of the equation is $r=0$ and when we evaluate the function at $r=1$ we get the value -0.022. Since our function is continuous, the only zero of our function is at 0 and our function is negative for $r>0$, we get that the expression is always true for $r>0$.

5.2 Numerical solution

In order to solve a differential equation, Matlab provides already implemented solvers. You can quite easily use them in a standard way, depending of the

structure of your differential equation. Even though our differential equation is a special case of a delay differential equation, Matlab can solve numerically our differential equation in a quite easy way. First of all, we need to rearrange our differential equation.

$$\begin{aligned} -2rF'\left(\frac{r}{2}\right)(1+F(r)) &= 4F\left(\frac{r}{2}\right)(1-F(r)) \\ F'\left(\frac{r}{2}\right) &= 4F\left(\frac{r}{2}\right)\frac{1-F(r)}{1+F(r)} \\ F'\left(\frac{r}{2}\right) &= -2F\left(\frac{r}{2}\right)\frac{1-F(r)}{r(1+F(r))}. \end{aligned} \tag{16}$$

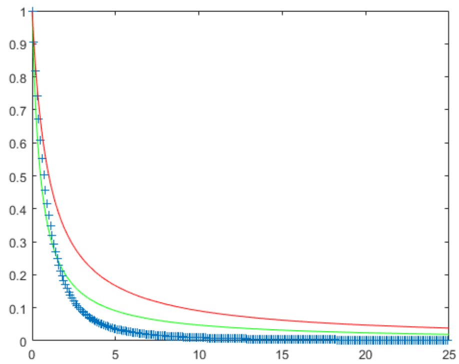
In our case, it is important to notice, that we are dividing by r. However, our initial condition states that $F(0) = 1$. If we would feed Matlab with the usual initial condition $F(0) = 1$, it would right away start our computation by outputting **NaN** (not a number), since we would divide by 0 during our first run. Hence, we need to somehow avoid this problem. The most easiest way (and which in most cases works perfectly fine) is to simply use approximated values for the initial condition. So, instead of taking 0 one would use $1e{-}8 = 10^{-8}$, which is close enough to 0 for our purpose. However, if you try to use $F(10^{-8}) = 1$ as an initial condition, you will at each step 1 as the result for the current step. This is largely caused by the constellation of your differential equation. To avoid this problem, one simply does the same trick as before, but by approaching 1 from below. Hence, we get the initial condition $F(10^{-8}) = 1 - 10^{-8}$, which is good enough for our situation. Notice, that we cannot approach 1 from above, because our initial condition is derived from the density property and hence we cannot get bigger values than 1. Once we have made these observations, we can quite easily solve our differential equation numerically by

```
% solve differential equation and store results in the vector (t,y)
[t,y] = ode45(@(t,y) -2*y*(1-y)/(t*(1+y)) , [1e-8:.1:25], 1-1e-8);
% plot the result and mark computated values by a + sign
plot(t,y,'+')
```

To solve our equation, we used the solver **ode45**, which is Matlab's standard solver for ordinary differential equations. Further, we do not need to specify that our function is evaluated at times t and $\frac{t}{2}$, since we can solve this differential equation numerically without making this distinction. It remains to explain what **[1e-8:.1:25]** stands for: it is our evaluation inverval. The algorithm to compute our numerical solutions starts at **1e-8** up to **25** by calculating a value for every step of magnitude **0.1**. To output the result, one can simply use the following command

```
% output solution
[t,y]
```

Further, we can plot our result



and compare it to the plot of the function $\frac{1}{1+x}$ in red and the function $\frac{0.5}{0.5+x}$ in green. From the previous computation, we know that $\frac{a}{a+x}$ is a solution for our problem. By comparing it to our numerical solution, we can conclude that we got a satisfying numerical approach of the solution.

5.3 Existence of other solutions

By the numerical approach we could find the most interesting solution of our differential equation: a solution of the form $\frac{a}{a+x}$, which then after the inverse Laplace transform yields the exponential distribution. However, we have also found another solutions in this paper, namely, the Dirac measure. So, where is that solution? Why could we not find it with our numerical approach? The reason is actually simple: the numerical approach only enables us to find one single solutions and not all together. In order to find different solutions, one might need to add more restrictions and play a little with the parameters. So, how can we achieve this? The one thing we know for sure is that every possible solutions will go through the point $(0, 1)$. Consequently, we could try to vary the derivative at this specific point. In order to do this, we are going to change our differential equation a little bit, namely, in the following way

$$F'(\frac{r}{2}) = \begin{cases} -2F\left(\frac{r}{2}\right)\frac{1-F(r)}{r(1+F(r))}, & \text{if } r > 0 \\ a, & \text{if } r = 0, \end{cases}$$

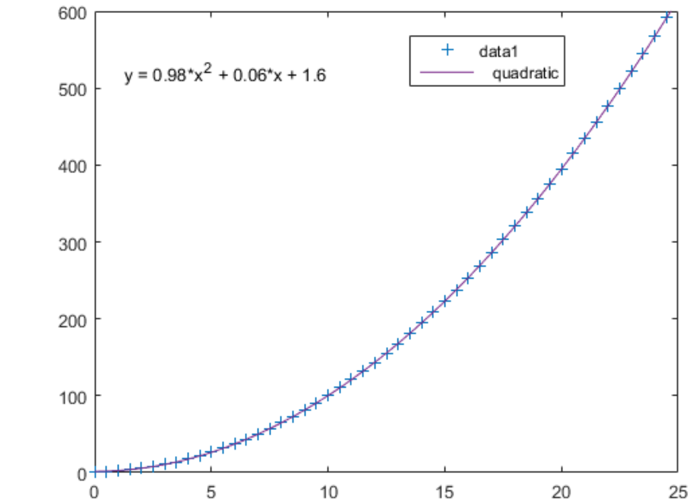
where a is some constant value. This way, we can play a little with our differential equation and we can vary the derivative at the initial condition. Since we are only changing the value at one specific point, we do not need

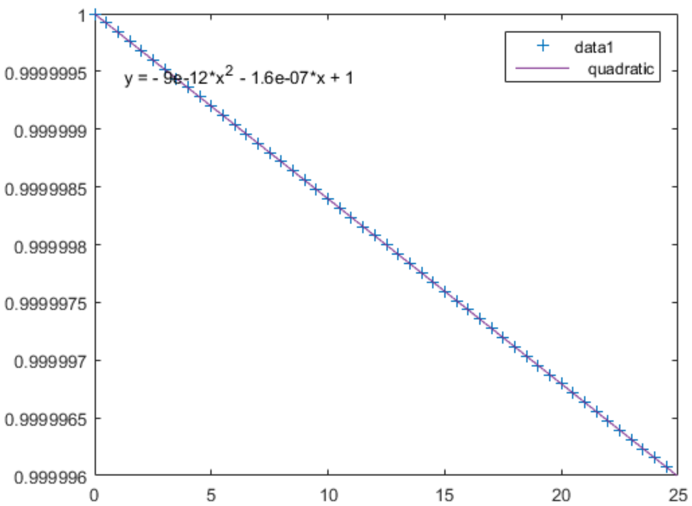
to define a function depending on t. It is sufficient to only use constant functions. For the solution, which we already found with our numerical approach, we immediately see that the derivative is negative at the initial condition. In fact, by varying the function by a negative constant at the initial condition, we are not going to find a new solution to our problem. In MATLAB, we can now solve our differential equation by using

```
% solve differential equation and store results in the vector (t,y)
[t,y]
= ode45(@(t,y) -2*y*(1-y)/(t*(1+y))*(t>1e-8)
        +(t==1e-8)*(1), [1e-8:.1:25], 1-1e-8);
% plot the result and mark computated values by a + sign
plot(t,y,'+')
```

where we defined a positive value for the derivative at the initial condition. Without loss of generality, we can fix the derivative to 1 at 0. For every other positive value, we will get the same result. The conditions $(t > 1e-8)$ and $(t == 1e-8)$ are Boolean conditions, which output a 1 if the condition is fulfilled and a 0 if the condition is not fulfilled. So, for every value for t, which is different from the initial condition, we get back our initial differential equation. After the plot, we can use a build-in MATLAB function to find an approximation of our result. For this, go to the tab **Tools**, then to **Basic Fitting** and choose the option **Show equation** after having selected the approximation you want. If we choose a quadratic approximation for our solution, we see that the existing explicit solution for our problem is actually a polynomial expression. Hence by previous computations, we have shown, that this gives us the Dirac measure as a solution. Further, if we choose to set the derivative to 0 at the initial condition, we just get the constant function $F(\frac{r}{2}) = 1$ as a result. Hence, again the Dirac measure. In this case it is important to not let you trick by the plot (see figure). This implies, that most probably, we have found all possible solutions for our differential equation.

MATLAB will give the following results





References

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